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A FAMILY OF AXISYMMETRIC VORTEX FLOWS WITH A SURFACE DISCONTINUITY OF THE BERNOULLI CONSTANT*

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One-parameter class of steady axisymmetric vortex flows of incompressible inviscid fluid with vorticity satisfying the Prandtl—Batchelor condition is considered. The Bernoulli constant becomes discontinuous and changes by a given quantity at the surface separating the external potential stream from the vortex flow region. Determination of the stream function is reduced to solving a system of two nonlinear integral equations for the boundary of the vortex flow region and of intensity of the vortex sheet contained in it. Results of numerical calculations are presented.

Existence of steady axisymmetric vortex rings similar to a circular vortex line and Hill's spherical vortex was established earlier in /1—3/, where flows of this type were described. A numerical solution linking these asymptotic results was obtained in /4/ for a set of vortex rings. Vortex flow inside the core of each ring is characterized by the constancy of the ratio of vorticity to the distance from the axis of symmetry, with the Bernoulli constant continuous in the stream. The introduced in /4/ parameter α (the dimensionless mean radius of the vortex ring core) which distinguishes the unique solution from the set ranges from zero (circular vortex) to $\sqrt{2}$ (Hill's spherical vortex).

The considered here set of vortex flows is characterized by parameter $\alpha > \sqrt{2}$ and the discontinuity of the Bernoulli constant at the boundary of vortex flow. One of the limit cases of this set of flows is also Hill's spherical vortex. The assumption of existence of a set of axisymmetric vortex flows with discontinuity of the Bernoulli constant adjoining Hill's vortex was expressed in /5/. Existence of a similar one-parameter class of plane vortex flows was established in /6/, where preliminary results of investigation of the axisymmetric class of vortex flows were also presented. Analysis of such flows is further refined below.

1. Consider a uniform at infinity steady axisymmetric flow of incompressible inviscid fluid, which within some bounded region Ω is vortical, and outside it potential. Let the vorticity of the vortex flow satisfy the Prandtl—Batchelor condition, i.e. in the cylindrical system of coordinates x, r, φ

$$\operatorname{rot} \mathbf{v} = (0,\ 0,\ Wr)$$

where v is the fluid velocity and W a constant. All quantities are dimensionless with the unperturbed stream velocity taken as the unit of velocity, and the maximum longitudinal dimension of the vortex flow region as the unit of length. The system of coordinates in the axial plane is shown in Fig.1. Let the Bernoulli constant become discontinuous of the stream surface S which is the boundary of region Ω. The Bernoulli integral and the continuity of pressure imply that then the condition

$$v_e^2 - v_i^2 = \Delta \tag{1.1}$$

must be satisfied on surface S. Here Δ is a constant equal to the doubled Bernoulli constant, and subscripts e and i relate to external and internal limit values on surface S.

For the Stokes stream function ψ in its conventional form

$$\mathbf{v} = \left(\frac{1}{r}\frac{\partial\psi}{\partial r},\ -\frac{1}{r}\frac{\partial\psi}{\partial x},\ 0\right) \tag{1.2}$$

we have on the basis of the above the following problem: determine for a given value of parameter $\Delta > 0$ parameter W, the vortex flow region boundary S, and the stream function ψ that satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} = \begin{cases} -Wr^2 & \text{inside } \Omega \\ 0 & \text{outside } \Omega \end{cases}$$

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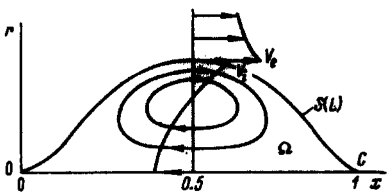

Fig.1

and the following conditions:

$$\psi \to {}^1/_2\, r^2 \quad \text{as} \quad x^2 + r^2 \to \infty \qquad (1.3)$$
$$\psi|_S = \text{const}; \quad |\nabla\psi|_e^2 - |\nabla\psi|_i^2 = r^2\Delta$$

where ∇ is the Hamiltonian, and the last condition follows from (1.1).

2. For an axisymmetric flow without azimuthal motion the vector potential is

$$\mathbf{A} = (0,\ 0,\ \psi / r) \qquad (2.1)$$

which for a given distribution of vorticity $\boldsymbol{\omega} = \text{rot}\,\mathbf{v}$ satisfies Poisson's equation and can, consequently, be written in the form

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\iiint \frac{\boldsymbol{\omega}'}{R}\, dV(\mathbf{r}'), \quad R = [(x - x')^2 + r^2 + r'^2 - 2rr'\cos\vartheta]^{1/2}, \quad \vartheta = \varphi - \varphi' \qquad (2.2)$$

where R is the distance between points $\mathbf{r}$ and $\mathbf{r}'$ of integration, and the integral is taken over the volume occupied by the fluid.

Discontinuity of the Bernoulli constant at the boundary of vortex flow means that S is a vortex surface of intensity $\omega = -(v_e - v_i)$. In conformity with (2.1) and (2.2) and allowance for the first of conditions (1.3) the stream function is of the form

$$\psi(x, r) = {}^1/_2 r^2 + r(A_1 + A_2), \quad A_1 = \frac{W}{4\pi}\iiint_{\Omega} \frac{r'\cos\vartheta}{R}\, d\Omega, \quad A_2 = \frac{1}{4\pi}\oiint_S \frac{\omega'\cos\vartheta}{R}\, dS \qquad (2.3)$$

In a Cartesian system of coordinates x, y, z with origin at point O (Fig.1) and the Oy-axis coinciding with the Or-axis of a cylindrical system of coordinates x, r, φ the vector potential coordinates is the direction of the Oz-axis are at $\varphi = 0$

$$A_1^\circ = \frac{W}{4\pi}\iiint_{\Omega} \frac{r'\cos\varphi'}{R}\, d\Omega, \quad A_2^\circ = \frac{1}{4\pi}\oiint_S \frac{\omega'\cos\varphi'}{R}\, dS$$

and represent the potentials of continuous distribution and of the simple layer, respectively. Let the curvature of surface S be finite at all points, except points O and C (Fig.1). When $\varphi = 0$ and $y = r$, functions A_1 and A_2 coincide, respectively, with A_1° and A_2°, hence, using the known properties of potentials and taking into account the axial symmetry of the flow, it is possible to conclude that functions A_1 and A_2, and the first order partial derivatives of function A_1 are continuous at transition through surface S, while the derivatives of function A_2 become discontinuous at that surface. Using the relations of limit values of derivatives of the simple layer potential taken along the normal to the surface, we find that at all points of surface S, except O and C,

$$\frac{\partial A_2}{\partial n_e} + \frac{\partial A_2}{\partial n_i} = 2\frac{\partial A_2}{\partial n_0} = \frac{1}{2\pi}\oiint_S \frac{\omega'\cos\vartheta\cos\gamma}{R^2}\, dS$$

where γ is the angle between the outer normal at the considered point of the surface and the radius vector drawn from that point to the integration point.

Assuming that surface S is the stream surface, i.e. that the stream function on it is constant, from (1.2) and (2.3) we obtain

$$v_{e,i} = \left[\frac{\partial A_1}{\partial n} + \frac{\partial A_2}{\partial n} + \left(1 + \frac{A_1 + A_2}{r}\right) n_r\right]_{e,i}$$

where n_r is the projection of the outer normal to surface S on the Or-axis.

The last of conditions (1.3) assumes the form

$$\Delta = -\omega(v_e + v_i) = -2\omega\left[\frac{\partial A_1}{\partial n} + \frac{\partial A_2}{\partial n_0} + \left(1 + \frac{A_1 + A_2}{r}\right) n_r\right], \quad (x, r) \in L' \qquad (2.4)$$

where L' is the curve L (defined by the intersection of surface S with the axial plane $\varphi = \text{const}$) with the exclusion of points O and C.

The second of conditions (1.3) yield the equation

$$\tfrac{1}{2} r + A_1(x, r) + A_2(x, r) = 0, \quad (x, r) \in L \tag{2.5}$$

We have, thus, obtained a system of Eqs. (2.4) and (2.5) in unknown functions $\omega(x)$ and $f(x)$, ($r = f(x)$ is the equation of curve L), which contains the unknown parameter W.

Let us consider the boundary conditions that must be satisfied by functions f and ω. Boundary of the vortex flow at its intersection points O and C with the axis of symmetry when $\Delta \neq 0$ must have cusps, with zero angles of pointing, i.e. the derivative of the function must be zero at these points, since otherwise the internal and external limit flow velocities would vanish at these points, which would contradict condition (1.1). Since at the inner side of the cusp surface S the velocity vanishes, hence it follows from (1.1) that when $\Delta > 0$, $v_i = 0$ at points O and C, i.e the cusps are directed outward from surface S and $v_e = \sqrt{\Delta}$. We thus have

$$f(0) = f(1) = f_x(0) = f_x(1) = 0, \quad \omega(0) = \omega(1) = -\sqrt{\Delta}$$

Expressions for parameter W in terms of f and ω are obtained from the condition $v_e(0) = \sqrt{\Delta}$. For this we consider at some point N of the axis of symmetry the velocity dependent on the vortex surface S

$$v(N) = \frac{1}{r} \frac{\partial (r A_2)}{\partial r}\Big|_N \tag{2.6}$$

It can be shown (by integrating by parts) that

$$\frac{1}{r} \int_0^{2\pi} \frac{\cos \vartheta}{R} d\vartheta = \int_0^{2\pi} \frac{r' \sin^2 \vartheta}{R^3} d\vartheta$$

Hence velocity (2.6), after differentiation with respect to r and integration with respect to angle ϑ, assumes the form

$$v(N) = \frac{1}{2} \int_L \frac{\omega' r'^2}{R^3} dl \equiv I(\sigma) \quad (R^2 = (x' - \sigma)^2 + r'^2)$$

We shall prove that when the integral

$$I(0) = \frac{1}{2} \int_L \frac{\omega' r'^2}{R_0^3} dl \quad (R_0^2 = x'^2 + r'^2)$$

is convergent, then as point N is approaching point O from outside the cusp

$$\lim I(\sigma) = I(0), \quad \sigma \to 0 \tag{2.7}$$

Convergence of the integral $I(0)$ implies that for any arbitrary $\delta > 0$ there exists an ε-neighborhood of point O for which $| I(0) |_{L_\varepsilon} | < \delta / 3$, where

$$I(0) |_{L_\varepsilon} = \frac{1}{2} \int_{L_\varepsilon} \frac{\omega' r'^2}{R_0^3} dl$$

and L_ε is the part of curve L that belongs to the ε-neighborhood of point O. By virtue of the integrand continuity we have for fairly small σ

$$| I(\sigma) |_{L-L_\varepsilon} - I(0) |_{L-L_\varepsilon} | < \delta / 3$$

Since point N is outside the cusp of surface S, we have in the small neighborhood of point O, $R \geqslant R_0$ and, consequently, $| I(\sigma) |_{L_\varepsilon} | \leqslant | I(0) |_{L_\varepsilon} |$. As the result we have

$$| I(\sigma) - I(0) | = | I(\sigma) |_{L-L_\varepsilon} - I(0) |_{L-L_\varepsilon} + I(\sigma) |_{L_\varepsilon} - I(0) |_{L_\varepsilon} | \leqslant | I(\sigma) |_{L-L_\varepsilon} - I(0) |_{L-L_\varepsilon} | + 2 | I(0) |_{L_\varepsilon} | < \delta$$

which proves (2.7).

Note that convergence of the improper integral $I(0)$ is not only the sufficient but, also, the necessary condition of existence of finite limits of velocity at point O. It can be shown that when integral $I(0)$ is divergent, the velocity $v(N)$ infinitely increases as point N approaches zero.

With allowance for (2.7) we can represent the external flow velocity at point O as

$$v_e(O) = \sqrt{\Delta} = 1 - \frac{1}{2} \int_L \frac{\omega' r'^2}{R_0^3} dl - \frac{W}{4\pi} \iiint_\Omega \frac{r'^2}{R_0^3} d\Omega \tag{2.8}$$

3. After some transformations, Eqs. (2.4), (2.5), and (2.8) assume the form

$$\frac{\Delta}{\omega}=-\frac{2}{\sqrt{1+r_x^2}}\Big\{1+\frac{1}{4\pi r}\oint\!\!\oint_S \frac{\omega'\cos\vartheta}{R}\,dS+\frac{1}{4\pi}\oint\!\!\oint_S \omega'\cos\vartheta\,[r_x(x-x')+r'\cos\vartheta-r]\,R^{-3}dS+ \tag{3.1}$$
$$\frac{W}{4\pi}\iiint_\Omega r'[r'-r\cos\vartheta+r_x\cos\vartheta\,(x-x')]R^{-3}d\Omega\Big\}$$

$$r=-\frac{W}{2\pi}\iiint_\Omega \frac{r'\cos\vartheta}{R}\,d\Omega-\frac{1}{2\pi}\oint\!\!\oint_S \frac{\omega'\cos\vartheta}{R}\,dS \tag{3.2}$$

$$\sqrt{\Delta}=1-\frac{W}{2}+\frac{W}{2}\int_0^1 \frac{r^2+2x^2}{(r^2+x^2)^{1/2}}\,dx+\frac{1}{2}\int_0^1 \frac{\omega r^2(1+r_x^2)^{1/2}}{(r^2+x^2)^{3/2}}\,dx \tag{3.3}$$

$$(r=f(x),\quad r'=f(x'))$$

When $\Delta=0$ the solution of this system is Hill's spherical vortex

$$f(x)=[1/4-(x-1/2)^2]^{1/2},\quad \omega(x)\equiv 0,\quad W=-30$$
$$\psi(x,r)=\begin{cases}-3r^2(1/4-\rho^2), & \rho\leqslant 1/2\\ 1/2\,r^2(1-1/8\,\rho^{-3}), & \rho\geqslant 1/2\end{cases}\qquad (\rho^2=(x-1/2)^2+r^2)$$

The system of Eqs.(3.1)—(3.3) was solved numerically for $\Delta>0$ using iteration method.

Subsequent approximations of functions f and ω were derived from Eqs.(3.1) and (3.2) whose right-hand sides were calculated using preceding approximations, and parameter W was determined from Eq.(3.3). Functions f and ω were assumed axisymmetric relative to the straight line $x=1/2$. For the computation of integrals in terms of angle ϑ we used the known formulas in which velocities and stream functions of the vortex ring are expressed in terms of complete elliptic integrals of the first and second kind. Singularities at $(x',r')=(x,r)$ were eliminated by analytic integration over a small neighborhood of point (x,r). Cubic splines were used for interpolating functions f and ω. To reduce the computation time each spline node was provided with its own integration grid so as to have at its nodes the parameter of complete elliptic integrals assume a priori specified values. This had enabled us to use tables of complete elliptic integrals in integration.

At the beginning with $\Delta=0{,}01$ we obtained a flow close to Hill's vortex. Subsequent solutions were obtained using previous solutions, changed in the required direction, as the zero approximations. Computations were discontinued when five significant digits were the same in two consecutive approximations. For $\Delta\leqslant 0{,}4$ 20—30 iterations were necessary for this. Since the number of necessary iterations increases with increase of parameter Δ and ensuing growth of required computer time, no solutions were sought for $\Delta>0{,}6$. Boundary of the region of vortex flow f and the vortex surface intensity ω calculated for several values of parameter Δ are shown in Fig.2, where the numbers relate to the values of Δ tabulated as follows:

№	Δ	W	r^*	α
1	0	—30	0,5	2
2	0,05	—40,5	0,4255	1,443
3	0,2	—59,0	0,3380	1,533
4	0,6	—136,3	0,1875	1,931

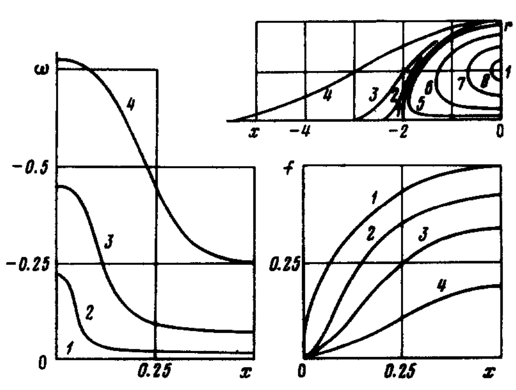

Fig.2

Maximum values r_* of functions f and of parameter α introduced in /4/ in the investigation of set of vortex rings using formula $S_0=\pi r_0^2\alpha^2$, where S_0 is the area of axial cross section of the fluid vortex motion and r_0 is the mean radius of the region cross section at the place of its maximum bulging, are also shown there. For vortex flows considered here $r_0=r_*/2$.

It will be seen that as the jump of the Bernoulli constant increases parameter α monotonically, increases from $\sqrt{2}$. In the case of one-parameter set of vortex rings /4/ parameter α changes from zero (circular vortex line) to $\sqrt{2}$ (Hill's spherical vortex).

Thus the set of vortex flows with discontinuity of the Bernoulli constant joins the set of vortex rings when $\alpha=\sqrt{2}$.

Boundaries of axial cross sections of the vortex flow regions are also shown in Fig.2 for several values of parameter α. Curves 5–8 correspond to $\alpha = 1.35, 1, 0.6, 0.2$. The linear dimensions are normalized with respect to r_0. The circular vortex line passes through point $x = 0, r = 1$. For $0 < \alpha < \sqrt{2}$ the vortex flow region is of toroidal form, when $\alpha > \sqrt{2}$ it adjoins the axis of symmetry, and the region boundary at intersection points with the latter has singular points. These two sets are linked by Hill's spherical vortex $(\alpha = \sqrt{2})$ whose vortex flow region is attached to the axis of symmetry, but owing to the absence of discontinuity of the Bernoulli constant $(\Delta = 0)$, it has a smooth boundary.

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